MULTIPLE DECREMENT THEORY

Griffith Feeney
East-West Population Institute
The East-West Center
Honolulu, Hawaii 96822
June 1974

1. Introduction

When vaccination was discovered in the eighteenth century contemporaries raised the question of the effect on the level of mortality of the elimination of smallpox as a cause of death. Since persons who would otherwise have died of smallpox would evidently die of other causes, deaths due to other causes would increase. What would be the overall effect of eliminating smallpox? For example, how much would the expectation of lift be increased? What proportion of persons would survive to various ages? In general, what would the various life table statistics look like after the elimination? Since all life table statistics are determined, to within a close approximation, by the probabilities of death for each year of age, the general problem is reduced to this. How would the probability of death over a given age interval change as the result of eliminating a particular cause of death?

The answer to this question turns out to be more difficult than one might suspect. The essential difficulty may be illustrated by considering a situation in which a group of N persons who attain age x experiences D deaths before reaching age $x+n$ of which D_s are due to smallpox. We would in this case estimate the probability of death by causes other than smallpox by $(D-D_s)/M$. Now, at first glance, it might seem reasonable that this would be the probability of death from all remaining causes upon the elimination of smallpox. Upon further reflection, however, one suspects that this would give too low an estimate of the probability of death. Why? Because the D_s persons who do not die of smallpox are still subject to death by other causes, and it is reasonable to suppose that some of them will die. Indeed, in the special case of $n = \infty$ they must die, else we be granting immortality to persons who would previously have died of smallpox. One suspects, then, that the probability of death once smallpox has been eliminated will be greater than $(D - D_s)/N$. How a much greater? That is the difficult question which gave rise to what is known as the theory of multiple decrements. This theory was developed in the late nineteenth century by actuaries and has recently been recognized as a special branch of the theory of stochastic processes. Similar problems arise in other areas, and the phrase "competing risks" is some-times used in place of "multiple decrements."

2. Probabilities of Death by Different Causes

It is conventional in discussing probabilities of death to refer to "survival to age x" and to use the notation $s(x)$. One might of course refer to "death after age x," for this means exactly the same thing, and use the notation $d(x)$. So long as causes of death are not distinguished, there is no reason to prefer either form. When considering causes of death, however, the phrasing in terms of death is distinctly superior, for the phrase "death by cause i after age x" is sensible whereas "survival to age x by cause i" is gibberish. Actuaries are apparently unmoved by this argument and persist in the unnatural notation. We shall not follow suit in these notes.

Probabilities of death by several causes may be expressed in terms of mathematical functions just as in the case where no causes of death are distinguished. (In the context of multiple decrement theory this is usually referred to as the "single decrement case.") One function is introduced for each cause of death. Consider

$$d_1(x) = d_2(x) = \begin{cases} 1/2 - x/200 & \text{if } x \le 100; \\ 0 & \text{otherwise.} \end{cases}$$

These functions may be regarded as giving the probabilities of death after age x by either of two different causes. In particular, $d_1(0)$ gives the probability of dying of the first cause, $d_2(0)$ the probability of dying from the second cause. There are evidently no other causes of death in this example, for $d_1(0) + d_2(0) = 1$. The probability of dying of the $i - th$ cause between age x and age $x + n$ is given by $d_i(x) - d_i(x + n), i = 1, 2$. The conditional probability of death by the $i - th$ cause between age x and age $x + n$ given survival to age x is given by $(d_i(r) - d_i(x + n))/d(a)$, where $d(x)$ is defined to be $d_1(x) + d_2(x)$, the probability of survival to age x. These expressions are derived just as in the single decrement case.

3. Forces of Mortality

The key to the solution of the multiple decrement problem is the concept of the "force of mortality" due to a particular cause of death. Although the force of mortality is an intrinsically probabilistic concept, the conventional actuarial treatment is nonprobabilistic and accordingly somewhat awkward. A good exposition of the actuarial perspective, which recognizes the awkwardness, is given in [6]. The probabilistic approach has been developed in a series of papers by Hoem, of which [2] and [3] are particularly relevant here. Consider the conditional probability of death by cause i between age x and age $a + n$ given survival to age a. This probability will approach zero as the length of the age interval is taken smaller and smaller. Suppose however that this conditional probability is divided by h, so that we are referring to a probability of death per unit time. This quantity approaches a positive limiting value as h approaches zero, and this limiting value is called the force of mortality at age x due to cause i. Applying this definition we find that

$$\lim_{h \to 0} \frac{d_i(x) - d(x + n)}{hd(x)} = \frac{-1}{d(x)} \lim_{h \to 0} \frac{d_i(x + h) - d_i(x)}{h} = \frac{-d_i'(x)}{d(x)}$$

The force of mortality for cause i at age x is denoted by $\mu(x)$.

Forces of mortality are additive in the sense that the force for any two causes of death taken together as a single cause equals the sum of the forces for the individual causes. By definition, the force for cause i and cause j regarded as a single cause equals

$$\lim_{h \to 0} \frac{(d_i(x) + d_j(x)) - (d_i(x + h) + d_j(x + h))}{hd(x)}$$

But this equals

$$\lim_{h \to 0} \frac{d_i(x) - d_i(x + h)}{hd(x)} + \lim_{h \to 0} \frac{d_j(x) - d_j(x + h)}{hd(x)}$$

which in turn equals $\mu_i(x) + \mu_j(x)$. In particular, the force of mortality for all causes combined is the sum of the forces for all the individual causes. The overall force of mortality at age x is denoted by $\mu(x)$.

Forces of mortality are defined by the equations

$$\mu_i(x) = \frac{-d'i(x)}{d(x)}. \tag{3.1}$$

Suppose we are given the forces of mortality and wish to determine the probability functions di. How shall we proceed? We begin by solving the differential equation

$$\mu(x) = \frac{-d'(x)}{d(x)} \tag{3.2}$$

for the function $d(x)$. This yields

$$d(x) = exp\{-\int_0^x \mu(a)da\}$$

Now (3.1) may be written

$$\mu_i(x)d(x) = -d_i'(x).$$

Integrating both sides of this equation from x to L and taking the limit as L goes to infinity we find that

$$d_i(x) = \int_x^\infty \mu_i(a)d(a)da. \tag{3.3}$$

Equations (3.2) and (3.3) together give the desired expression of the probability functions d_i in terms of the force of mortality functions μ_i.

We have said that the force of mortality is an intrinsically probabilistic concept. Does it have a direct probabilistic interpretation? In the following sense, yes. It follows from the definition that ${}_hq_x^i$, the probability, for a person aged x, of death by cause i before age $x+h$, equals $\mu_i h + o(h)$ where $o(h)$ is a quantity which goes to zero as h goes to zero. (In fact, $o(h)/h$ goes to zero as h goes to zero, a fact which figures in the following paragraph.) Consequently the quantity $\mu_i h$ may be regarded as an approximation to ${}_hq_x^i$. In particular, taking $h = 1$, $\mu_i(x)$ approximates q_x^i. This is the easiest way to remember what the force of mortality means.

This interpretation of the force of mortality may be used to give a probabilistic derivation of equations (3.1, 3.2, 3.3). Let the force of mortality function μ_i for all causes be given and let $d(x,h)$ denote the probability of death between the ages x and $x+h$. By the definition of conditional probability, $d(x,h) = (\mu(x)h + o(h))d(x)$, and since $d(x,h) + d(x+h) = d(x)$, we find that $(\mu(x)h + o(h))d(x) + d(x+h) = d(x)$. Multiplying both sides of this equation by $1/hd(x)$ and rearranging terms yields

$$\mu(x) + \frac{o(h)}{h} = \frac{d(x) - d(x+h)}{hd(x)}.$$

Upon taking the limit of both sides of this equation we see that the left side goes to $\mu(x)$ and the right to $-d'(x)/d(x)$, hence we obtain equation (3.1). Equations (3.2) and (3.3) follow as before.

The above derivation is in fact a special case of the derivation of the Chapman-Kolmogorov equations for a finite state, continuous time Markov process. For an introductory account see [4].

4. Solution of the the Multiple Decrement Problem

We return now to the problem which motivated the development of all this mathematical machinery. That would be the effect on mortality of the elimination or a particular cause of death? We answer this question by assuming that the elimination of the cause of death in question means that the force of mortality from this cause will become zero at every age and that the force of mortality due to all other causes will remain unchanged. Given that the original forces of mortality are $\mu_1, \mu_2, ..., \mu_n$, and given that the $i-th$ cause of death is eliminated, the new force of mortality for all remaining causes will then be

$$\mu^* = \sum_{j \neq i} \mu_j$$

and the new probability of dying after age x from any cause will be

$$d_j^*(x) = \int_x^\infty d^*(x)\mu_j(x)dz$$

The probability of dying after age x of the $j-th$ cause, $j \neq i$, will be

$$d_j^*(x) = \int_x^\infty d^*(x)\mu_j(x)dx.$$

These three expressions define all probabilities of death under the hypothesis of the elimination of the $i-th$ cause of death.

Let us return to the example of section 2, in which there are two causes of death with probabilities given by

$$d_1(x) = d_2(x) = \begin{cases} 1/2 - x/100 & \text{if } x \leq 100; \\ 0 & \text{otherwise.} \end{cases}$$

The probability of death after age x by one or the other cause equals $d_1(x)+d_2(x) = 1-x/100$ if $x < 100$ and zero otherwise. The total force of mortality at age x is given by

$$\frac{-d'(x)}{d(x)} = \frac{1/100}{1-x/100} = (100-x)^{-1}$$

for $x < 100$. For $x > 100$ the force of mortality is undefined. Likewise, the forces of mortality for causes 1 and 2 are given by $\mu_1(x) = \mu_2(x) = (200-2x)^{-1}$.

If the second cause of death is eliminated in this example, the new force of mortality will imply be $\mu^*(x) = (200-2x)^{-1}$ and the new probability of death after age x will be given by

$$d^*(x) = exp\{-\int_0^x (200-2z)^{-1}dz\}.$$

This expression may be evaluated directly, or one may observe that it equals

$$exp\{-\frac{1}{2}\int_0^x (100-x)^{-1}dz\} = \big[exp\{-\int_0^x (100-z)^{-1}dz\}\big]^{1/2}$$

$$= d(x)^{1/2}$$

$$= (1-x/100)^{1/2}$$

Using this expression we can answer any question involving the new probabilities of death. Consider for example the expectation of life at birth. It is given by

$$\int_0^{100} d^*(x)dx = \int_0^{100} (1-x/100)^{1/2}dx$$

$$= \big[-\frac{200}{3}(1-x/100)^{3/2}\big]_0^{100}$$

$$= 200/3$$

$$= 66.7 \text{ years}$$

By comparison, when both causes 1 and 2 were operative, the expectation of life at birth is 50 years. Observe that the elimination of one of two equal causes of death does not double the expectation of life.

It will be well to reflect for a moment on the nature of this solution to the multiple decrement problem. Does the theory allow us to predict how mortality would change if, to take a specific case, cardiovascular disease were eliminated as a cause of death in the United States? The answer is no. The mathematical theory given here provides only half the answer. The other half would have to come from an empirical (and perhaps also theoretical) study aimed at determining whether, if cardiovascular disease were eliminated, the forces of mortality for other causes would in fact remain the same. It is certainly possible that they might, as the above solution to the multiple decrement problem assumes. However it is also conceivable that the elimination might change the interaction of the organism and the environment in such a way as to necessarily result in changes in the force of mortality due to the remaining causes, and in this case the above theory would be inadequate.

5. Cause of Death Life Tables

A multiple decrement life table is a life table in which the numbers of deaths in each age interval are distributed according to the cause of death. Since the number of columns in a multiple decrement table can easily become unmanageably large, it is conventional to present the data in the form of two tables, one showing the ordinary life table for all causes of death combined, the second indicating the incidence of the various causes of death. The latter table may give either the distribution by cause of deaths occurring in each age interval, or the distribution by cause of deaths occurring above specified ages. An extensive, critical, and international set of multiple decrement life tables is given in [5].

Multiple decrement tables are often accompanied by what are referred to as associated single decrement tables for particular causes of death. The associated single decrement table for a particular cause is the table which would result, according to the above theory, if this cause of death were eliminated. Because of the redundancy of the various columns of the life table, and because of the tremendous volume of numbers which arise, especially when many causes of death are considered, it is customary to give only a single column of the associated single decrement tables, usually the l_x column.

It was suggested in the introduction to these notes that the probability of death, in the presence of smallpox, by causes other than smallpox, would be lower than the probability of death by all remaining causes after the elimination of smallpox. One obvious application of the theory is to answer the question: how much lower? How important is it to invoke the theory in making numerical estimates? To answer this question specifically for smallpox would involve data collection and calculation that would take us too far afield here However the general point may be taken up using examples taken from the tables given in [5]. For Australian males aged 60 in 1911, for example, the probability of death by age 65 from some cause was 0.l44. The probability of death by causes other than cardiovascular diseases was 0.0827, 42 percent less than the probability of death by some cause. However the probability of death by some cause with cardiovascular disease eliminated equals 0.104, only 28 percent less than the overall death probability in the presence of cardiovascular diseases.

6. References

[1] Chiang, Chin Long. *Introduction to Stochastic Processes in Biostatistics.* New York: John Wiley and Sons, Inc. 1968.

[2] Hoem, Jan M. Purged and partial Markov chains. *Skandinavisk Aktuarietidskrift* **52**(3-4), 1969, 147-155.

[3] —. Point estimation of forces of transition in demographic models. *Journal of the Royal Statistical Society, Series B* **33**(2), 1971, 275-289.

[4] Parzen, Emanuel. *Stochastic Processes.* San Francisco: Holden-Day Inc. 1962.

[5] Preston, Samuel H., Nathan Keyfitz, and Robert Schoen. *Causes of Death: Life Tables for National Populations.* Studies in Population. New York: Seminar Press 1972.

[6] Smith, F. C. The force of mortality function. *The American Mathematical Monthly* **55**(5), May 1948, 277-284.

7. Acknowledgement and Version Note

Thanks to Basia Zaba for spotting several typographical errors in the original paper. Originally drafted June 1974. This corrected (but unrevised) version produced July 1999.

Hooray for TEX!